Lecture 1

Sunday, 21 August 2022 7:07 PM

Game Theory acc. to Roger Myerson:

"the study of mathematical models of conflict & cooperation between intelligent & rational decision-makers"

rational : if an agent has 2 choices, A & B, and it believes that A is better for it, then it must choose A. (Both simple & controversial; what does better actually mean?)

intelligent : agent can determine which options are better / worse for it.

mathematical model : a mathematically-defined abstraction of reality. Should be both simple enough to be analysed, & complex enough to capture the important characteristics of reality.
eg. Turing machines, weather models, models for Covid.

AGT, not GT :

Game theory is studied in various fields, including economics & OR. We will be taking a computation-oriented perspective, in particular focussing on:

- computation of equilibria & other solution concepts.
- approximation algorithms

We will NOT be focussing on:

- dynamic games, repeated games, incomplete information, ...

Game 1: Prisoner's Dilemma

2 small-time thieves are caught at the scene of a bank robbery. Police offers each thief a choice: they can cooperate (C) with the police, or don't cooperate (DC).
For Prisoner 1 (P1):

- if P1 & P2 cooperate : both get 4 years in prison.
- if P1 cooperates, P2 doesn't : P1 gets 1 year, P2 gets 6 yrs
- if P1 doesn't cooperate, P2 does : P1 gets 6 yrs, P2 gets 1 year
- if both don't cooperate : both get 2 years.

We can write this as follows:

P2

P1	C	DC
C	4 \ 4	1 \ 6
DC	6 \ 1	2 \ 2

A \ B ; A → P1's sentence, B → P2's sentence

We make the following assumptions in our modelling:

1. these numbers reflect the "unhappiness", or cost, of each agent. Thus if both don't cooperate, both get 2 units of "unhappiness".
2. both agents / players are rational
3. this is a simultaneous move game. i.e., P1 cannot wait for P2 to choose first, they choose at the same time
4. this is a single-shot game. They play only once, get the costs, and the game (and their choices) never affect them again.

Fix P1. No matter what P2 does, P1's cost is less if it chooses to cooperate. Hence, since P1 is rational, it must choose to cooperate. Similarly for P2.
(we say C is a "dominant strategy" for P1, and similarly for P2).
Hence, since P1 & P2 are both rational, they must choose C, and get 4 years each.
Note the discrepancy: if they both chose DC, they would only get 2 years. But rationality forces both to choose C.
Thus rationality is in this sense very short-sighted / selfish

Notation:

3 elements of a (single-shot, simultaneous-move) game:

(i) set N of n players, $\{1, 2, \ldots, n\} =: [n]$

(ii) for each player $i \in N$, a set S_i of strategies
$s_i \in S_i$ is a particular strategy for player i.
- $s = (s_1, s_2, \ldots, s_n)$ is a strategy tuple, or strategy profile, denoting a strategy for each player
- $S = S_1 \times S_2 \times \ldots \times S_n$ is the set of all strategy tuples

(iii) for each player $i \in N$, $u_i : S \to \mathbb{R}$ is a utility fn. (sometimes we may use c_i to denote a cost fn.) giving a value for player i for each strategy profile.

Sometimes, we may fix player i, and to differentiate it's strategy, use $s = (s_i, s_{-i})$
s_{-i}: strategy tuple for all players except i
similarly $S_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$

Defn (Strongly dominant strategy): For player i, $s_i^* \in S_i$ is a (strongly) dominant strategy if, for all $s_{-i} \in S_{-i}$, and all $s_i' \neq s_i^*$, $s_i' \in S_i$,

$$u_i(s_i^*, s_{-i}) > u_i(s_i', s_{-i})$$

i.e., no matter what the other players do, player i maximizes utility by playing s_i^*
It follows that in such a game, a rational player must always play s_i^* (i.e., if a player has a dominant strategy, it must always play this).

Similarly, we can define dominated strategies:

Defn (Dominated Strategy): For player i, strategy s_i is (strictly) dominated by $s_i' \in S_i$ if, for all $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$$

i.e., playing s_i' is always (strictly) better than playing s_i.
Note that in this case, a rational player will never player s_i.
Note also the asymmetry in the two defns.; a dominated strategy is dominated by another strategy, while a dominant strategy is better than all other strategies.

Game 2: IRDS game

P2

P1	L	M	R
U	4 \ 3	5 \ 1	6 \ 2
M	2 \ 1	8 \ 4	3 \ 6
D	3 \ 0	9 \ 6	2 \ 8

Can check that neither player has a dominant strategy.
We make one more assumption:
each player's utility & rationality are common knowledge
i.e., P1 knows P2's utilities & that P2 is rational.
P2 knows that P1 knows P2's utilities & rationality
P1 knows that P2 knows that P1 knows ...
Then, for P2, M is dominated by R.
hence, P2 never plays M
P1 knows that P2 never plays M.
In the reduced game, where P2 chooses b/w L & R,
U is a dominant strategy for P1.
Hence, assuming rationality & common knowledge,
P1 plays U & P2 plays L.
This procedure is known as Iterated Removal of Dominated Strategies (IRDS).

```
For all i ∈ N, let S_i' ← S_i , S' := S_1' × ... × S_n'
Flag ← True
while (Flag) {
    Flag ← False
    For all i ∈ N {
        if ∃ s_i, s_i' ∈ S_i' s.t., ∀ s_-i ∈ S_-i', u_i(s_i, s_-i) < u_i(s_i', s_-i) {
            S_i' ← S_i' \ {s_i}
            Flag ← True
        }
    }
}
```

Q1 The ½-average game
$n = 10$ players, $S_i = \{0, 1, \ldots, 100\}$ $\forall$ players i.

$$c_i(s_i, s_{-i}) = \left| \frac{1}{20} \sum_{j=1}^{10} s_j - s_i \right|$$

i.e., each player tries to get close to ½-average.
w/ above assumptions, what does IRDS return?

Q2 What if above each player tried to get as far as possible from ½-average?
i.e.,

$$u_i(s_i, s_{-i}) = \left| \frac{1}{20} \sum_{j=1}^{10} s_j - s_i \right| \quad ?$$

Game 3: The Canteen Game

B

A	EC	WC
East Canteen	7 \ 4	3 \ 3
W. Canteen	0 \ 0	4 \ 7

In this game no player has a dominated strategy.
In such a case, we consider "rationality in hindsight": did each player play the best strategy against the others?

Defn. (Pure Strategy Nash Equilibrium): Strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a PSNE (or just a Nash Equilibrium) if, for each player i, & each strategy $s_i' \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$$

That is, each player chose the best possible strategy, keeping fixed the strategies of the other players.

Can check : In the canteen game, both (EC, EC) & (WC, WC) are NE. (EC, WC) and (WC, EC) are not NE.

Q3 Let $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ be a NE in a game, and let $S' = (S_1', \ldots, S_n')$ be the reduced game obtained by applying IRDS to the game.
Show that $s_i^* \in S_i'$ for each player $i \in N$.